Poisson Distribution

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Overview

- The Poisson (pwa - son) Distribution is used to determine the probability of some number of events occuring within a given span of time
- It can be derived from the Binomial Distribution in which a biased coin is flipped repeatedly
- In computer simulations, the random generation of Poisson distributed events can be simplified
- In a Poisson Process, events occur continuously and independently of each other with a variance equal to the mean

Permutation

- Ordered arrangement of n distinct objects
- n choices for first selection,
  n -1 choices for second selection,
  n - 2 choices for third selection,
  [...],
  1 choice for last selection
- n * (n - 1) * (n - 2) * [...] * 3 * 2 * 1
- n factorial
- n!

Permutation subset

- Choose ordered subset k of n distinct objects
- Example: choose just 3 of 7 distinct objects
- 7 first choices, 6 second, 5 third
- n * (n - 1) * [...] * (n - k + 1)
- Example: 7 * (7 - 1) * (7 - 3 + 1) = 7 * 6 * 5
- Same as dividing n factorial by (n - k) factorial:
  n * (n - 1) * [...] * (n - k + 1) * (n - k) * [...] * 1

  (n - k) * [...] * 1
- n! / (n - k)!
- Example: 7! / (7 - 3)! = 7! / 4! = 7 * 6 *5

Combinations vs. Permutations

- Permutations ordered
  abc, acb, bac, bca, cab, cba
- Combinations unordered: abc
- Choose unordered subset k of n distinct objects
- Example: choose 2 from set of 3 (abc), ordered
- $n! / (n - k)! = 3! / (3 - 2)! = 3! / 1! = 3 * 2 = 6$
- ab, ac, ba, bc, ca, cb
- Example: choose 2 from 3, unordered
- ab (ab and ba), ac (ac and ca), bc (bc and cb)
- Number of combinations same as permutations divided by number of ways to permute subset

Combinations

- To get number of combinations (unordered)
  - First calculate the number of ways we can create an ordered subset of length k from n distinct objects
  - Then divide it by the number of ways to order k
- [n! / (n - k)!] / k!
- n! / [(n - k)! * k!]
- n choose k $\binom{n}{k}$
- Example: 3 choose 2
- 3! / [(3 - 2)! * 2!] = 3! / [1! * 2!] = 3
- combinations of 2 from abc: ab, ac, bc

Random Combinations

- Urn with three white balls numbered 1, 2, and 3
- Randomly draw two and paint them black
  - 6 Permutations: 1, 2; 1, 3; 2, 1; 2, 3; 3, 1; 3, 2
  - 3 Combinations: $w_1b_2b_3$, $b_1w_2b_3$, $b_1b_2w_3$
- Randomly draw one and paint it black
  - 3 Permutations: 1; 2; 3
  - 3 Combinations: $w_1w_2b_3$, $w_1b_2w_3$, $b_1w_2w_3$
- Randomly draw three and paint them black
  - 6 Permutations but just one combination: $b_1b_2b_3$

Balls to Coins

- Urn with three coins numbered 1, 2, and 3
- Coins in the urn all start tails up
- Randomly drawn coins are placed heads up
- Randomly draw two and place heads up
  - 6 Permutations: 1, 2; 1, 3; 2, 1; 2, 3; 3, 1; 3, 2
  - 3 Combinations: $t_1h_2h_3$, $h_1t_2h_3$, $h_1h_2t_3$
- Same as white balls being painted black
- Replace 3 drawn coins with 1 coin flipped thrice

Coin Flips

- Flip a coin thrice
  - zero heads and three tails (ttt)
  - one head and two tails (tth, tht, htt)
  - two heads and one tail (thh, hth, hht)
  - three heads and zero tails (hhh)
- Calculate using combinations
  - n choose k: n! / [(n - k)! * k!]
  - 3!/[(3-0)!*0!]+3!/[(3-1)!*1!]+3!/[(3-2)!*2!]+3!/[(3-3)!*3!]
  - 1 + 3 + 3 + 1 = 8 different ways

Ordering

- Permutations ordered, combinations unordered
- To determine the number of ways we can flip a coin n times and get k heads, we use the formula for unordered combinations
- Strange because it looks like we are ordering
  - two heads and one tail (thh, hth, hht)
- What was combined going from balls to coins?
  - second head, third head: thh
  - third head, second head: thh

Fair and Biased Coins

- Probability of heads on fair coin: $p = 0.5$ (50%)
- Biased coins do not have 50% chance of heads
- Probability of head on biased coin: $p = 0.9$
- Probability of tail: $q = 1 - p = 1 - 0.9 = 0.1$
- Whether fair or biased, the number of ways of getting 2 heads in 3 coin flips is still the same: n choose k
- The difference is in the probability of getting 2 heads in 3 coin flips

Binomial Distribution

- Binomial = "two names", e.g., "heads" or "tails"
- Probability of two heads in three coin flips?
- Pr(thh) + Pr(hth) + Pr(hht)
- $q * p * p + p * q * p + p * p * q = 3 * p^2 * q^1$
- Multiply p for each head, q for each tail, then multiply by the number of combinations

$$Prob(K=k) = \binom{n}{k} * p^k * q^{(n-k)}$$

Binomial Distribution Example

$$Prob(Y=k)=\binom{n}{k}*p^{k}*q^{(n-k)}$$

- n = 3, k = 2, p = 0.9
- $(n! / [(n - k)! * k!]) * p^{k} * (1 - p)^{(n - k)}$
- $(3! / [(3 - 2)! * 2!]) * 0.9^{2} * 0.1^{(3 - 2)}$
- $(3! / [1! * 2!]) * 0.9^{2} * 0.1$
- 3 * 0.9 * 0.9 * 0.1 = 0.243 (or 24.3%)
- For a fair coin it would be 0.375 (or 37.5%)
- Biased coin, 3 heads: $1 * 0.9^{3} * 0.1^{0} = 72.9\%$

Flipping Coins Over Time

- Suppose we flip a very biased coin once a second
- If $p = 1 / 60$, we would expect to have one head and 59 tails each minute, on average
- In some minutes we might have more heads and in some minutes none at all
- To figure out the probability of getting a certain number of heads (k) in a minute, we could use the binomial distribution ($n = 60$, $p = 1 / 60$)
- We could add to figure out the cumulative probability of getting less than a number of heads in a minute, e.g., less than 3 heads: $Pr(k=0) + Pr(k=1) + Pr(k=2)$

Events Per Second

- Now suppose we flip a coin every millisecond
- To keep the same event rate, use a constant
- Define λ (lambda) in units of heads per second
- $\lambda = p * n$ = heads per flip * flips per second
- To increase n to 1,000 flips per second while keeping the event rate the same at
  $\lambda = 1 / 60$ heads per second,
  the probability p would need to be adjusted to
  $p = \lambda / n = (1 / 60) / 1{,}000 = 1 / 60{,}000$

In the Limit, Part 1

$$\lim_{n\to\infty}\left[\binom{n}{k}\cdot p^{k}\cdot q^{(n-k)}\right]$$

$$\lim_{n\to\infty}\left[\left(\frac{n!}{(n-k)!\cdot k!}\right)\cdot\left(\frac{\lambda}{n}\right)^{k}\cdot\left(1-\frac{\lambda}{n}\right)^{(n-k)}\right]$$

$$\frac{\lambda^{k}}{k!}\cdot\lim_{n\to\infty}\left[\left(\frac{n!}{(n-k)!}\right)\cdot\left(\frac{1}{n}\right)^{k}\cdot\left(1-\frac{\lambda}{n}\right)^{(n-k)}\right]$$

In the Limit, Part 2

$$\frac{\lambda^k}{k!} \cdot \lim_{n\to\infty} \left[\left(\frac{n!}{(n-k)!} \right) \cdot \left(\frac{1}{n} \right)^k \cdot \left(1 - \frac{\lambda}{n} \right)^{(n-k)} \right]$$

$$\frac{\lambda^k}{k!} \cdot \lim_{n\to\infty} \left[\left(\frac{n!}{(n-k)!} \right) \cdot \left(\frac{1}{n} \right)^k \cdot \left(\frac{n}{n-\lambda} \right)^k \cdot \left(1 - \frac{\lambda}{n} \right)^n \right]$$

$$\frac{\lambda^k}{k!} \cdot \lim_{n\to\infty} \left[\left(\frac{n!}{(n-k)!} \right) \cdot \left(\frac{1}{n-\lambda} \right)^k \cdot \left(1 - \frac{\lambda}{n} \right)^n \right]$$

Pascal's Triangle

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^{(n-k)} \cdot y^k$$

$$1$$

$$1 \cdot x + 1 \cdot y$$

$$1 \cdot x^2 + 2 \cdot x \cdot y + 1 \cdot y^2$$

$$\binom{0}{0} \cdot x^{(0-0)} \cdot y^0$$

$$\binom{1}{0} \cdot x^{(1-0)} \cdot y^0 + \binom{1}{1} \cdot x^{(1-1)} \cdot y^1$$

$$\binom{2}{0} \cdot x^{(2-0)} \cdot y^0 + \binom{2}{1} \cdot x^{(2-1)} \cdot y^1 + \binom{2}{2} \cdot x^{(2-2)} \cdot y^2$$

Pascal's Triangle Identity

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^{(n-k)} \cdot y^k$$

$$\left(1+\frac{\alpha}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot \left[\frac{\alpha}{n}\right]^k$$

Exponential Function Identity

$$\underset{n\to\infty}{limit}\left(1+\frac{\alpha}{n}\right)^{n}$$

$$\underset{n\to\infty}{limit}\sum_{k=0}^{n}\binom{n}{k}\cdot\frac{\alpha^{k}}{n^{k}}=\underset{n\to\infty}{limit}\sum_{k=0}^{n}\left[\frac{n!}{k!\cdot(n-k)!}\right]\cdot\frac{\alpha^{k}}{n^{k}}$$

$$\underset{n\to\infty}{limit}\sum_{k=0}^{n}\left[\frac{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-k+1)}{k!}\right]\cdot\frac{\alpha^{k}}{n^{k}}$$

$$\underset{n\to\infty}{limit}\sum_{k=0}^{n}\left[\frac{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-k+1)}{n^{k}}\right]\cdot\frac{\alpha^{k}}{k!}=\sum_{k=0}^{\infty}\frac{\alpha^{k}}{k!}=e^{\alpha}$$

In the Limit, Part 3

$$\frac{\lambda^k}{k!} \cdot \lim_{n \to \infty} \left[\left(\frac{n!}{(n-k)!} \right) \cdot \left(\frac{1}{n-\lambda} \right)^k \cdot \left(1 - \frac{\lambda}{n} \right)^n \right]$$

$$\lim_{n \to \infty} \left(1 + \frac{-\lambda}{n} \right)^n = e^{-\lambda}$$

$$\frac{\lambda^k}{k!} \cdot e^{-\lambda} \cdot \lim_{n \to \infty} \left[\frac{n \cdot (n-1) \cdot [\ldots] \cdot (n-k+1)}{(n-\lambda)^k} \right]$$

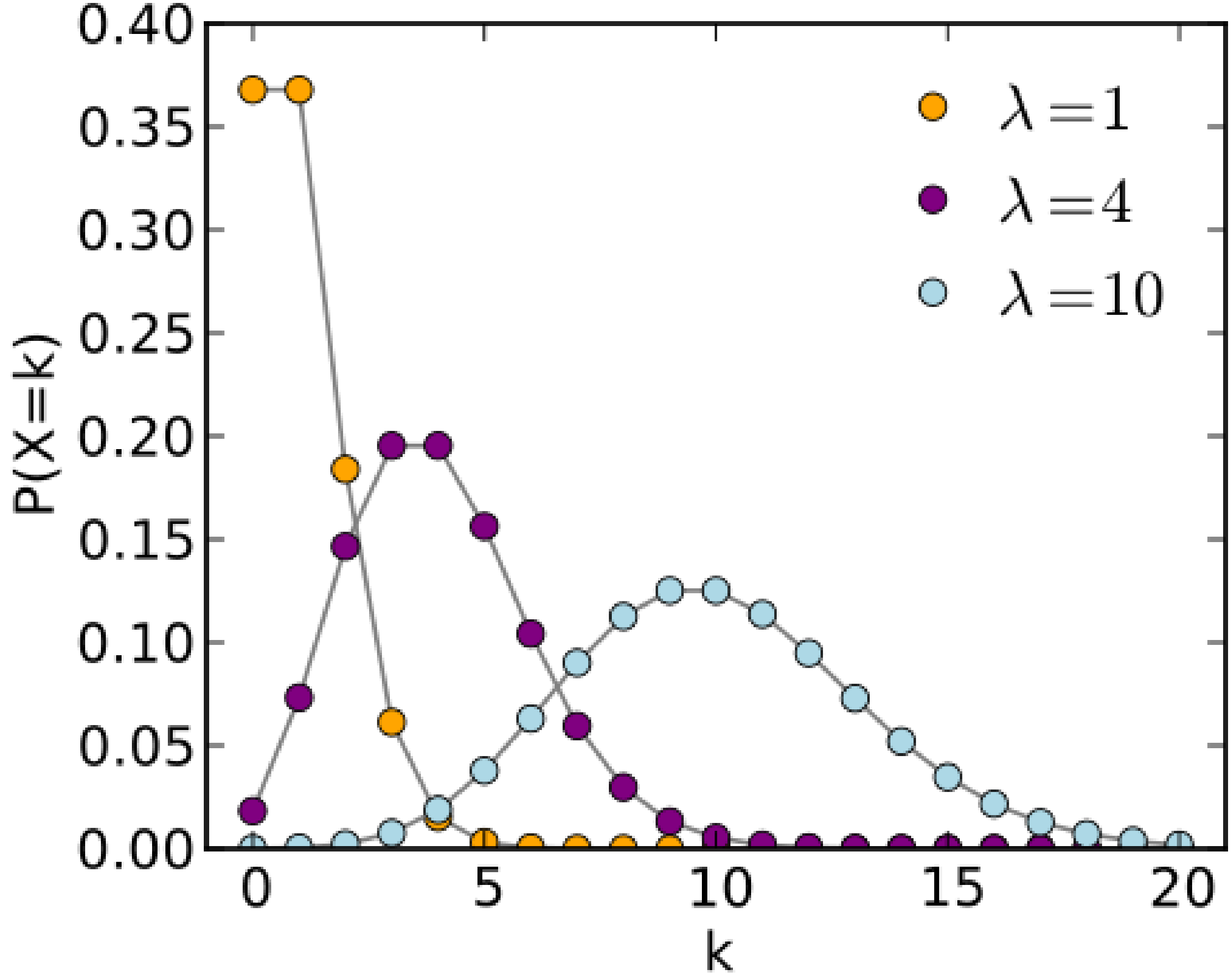
In the Limit, Part 4

$$\frac{\lambda^k}{k!} \cdot e^{-\lambda} \cdot \underset{n \to \infty}{limit} \left[\frac{n \cdot (n-1) \cdot [\ldots] \cdot (n-k+1)}{(n-\lambda)^k} \right]$$

$$\frac{\lambda^k}{k!} \cdot e^{-\lambda} \cdot \underset{n \to \infty}{limit} \left[\frac{n^k + [\ldots]}{n^k + [\ldots]} \right]$$

$$\frac{\lambda^k}{k!} \cdot e^{-\lambda}$$

Poisson distribution



Poisson Distribution

$$Prob(K=k)=\frac{\lambda^k}{k!}\cdot e^{-\lambda}$$

- If λ is 20 events per second, what is the probability that there will be one or more events in a twentieth of a second?

$$Prob(K=k)=\frac{(\lambda\cdot\Delta T)^k}{k!}\cdot e^{-\lambda\cdot\Delta T}$$

- ΔT = 0.05 seconds; λ * ΔT = 1
- Prob (K >= 1) = 1 - Prob (K = 0) = 1 - $e^{-\lambda*\Delta T}$ = 1 - e^{-1} = 1 - 0.367879 = 0.632121 = 63%

Poisson Simulation

$$Prob(K = k) = \frac{(\lambda \cdot \Delta T)^k}{k!} \cdot e^{-\lambda \cdot \Delta T}$$

- If your computer simulation is being updated with a very small time step, what probability do you assign a Poisson distributed event?
- Assume two or more events in the same very small time step improbable
- At 20 events per second and ΔT = 1 ms, Prob (K >= 1) = 1 - Prob (K = 0) = 1 - $e^{-\lambda * \Delta T}$ = 1 - $e^{-0.020}$ = 1 - 0.9801 = 0.0198 ~= 0.020
- NB: Prob (K >= 1) ~= $\lambda * \Delta T$ for small $\lambda * \Delta T$

Poisson Process

- Events occur continuously at some fixed rate
  - But are assumed not to occur simultaneously
- Events occur independently of each other
  - The property of memorylessness
- Examples: the count for a given unit of time of
  - Atoms that radioactively decay
  - Raindrops that land on a plate
  - Visitors to a website
  - Neural action potentials

Mean = Rate

$$E(K) = \sum_{k=0}^{\infty} k \cdot p(k, \lambda) = \sum_{k=0}^{\infty} k \cdot \left[\frac{\lambda^k}{k!} \cdot e^{-\lambda} \right]$$

$$\lambda \cdot e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^{(k-1)}}{(k-1)!}$$

$$\lambda \cdot e^{-\lambda} \cdot \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \, where \; j = k - 1$$

$$\lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda$$

Variance = Mean, Part 1

$$Var(K) = \sum_{k=0}^{\infty} (k-\lambda)^2 \cdot p(k,\lambda) = \sum_{k=0}^{\infty} (k^2 - 2\cdot k\cdot\lambda + \lambda^2)\cdot\left[\frac{\lambda^k}{k!}\cdot e^{-\lambda}\right]$$

$$\sum_{k=0}^{\infty} k^2\cdot\left[\frac{\lambda^k}{k!}\cdot e^{-\lambda}\right] - 2\cdot\lambda\cdot\sum_{k=0}^{\infty} k\cdot\left[\frac{\lambda^k}{k!}\cdot e^{-\lambda}\right] + \lambda^2$$

$$\sum_{k=0}^{\infty} k^2\cdot\left[\frac{\lambda^k}{k!}\cdot e^{-\lambda}\right] - 2\cdot\lambda\cdot\sum_{k=0}^{\infty} k\cdot\left[\frac{\lambda^k}{k!}\cdot e^{-\lambda}\right] + \lambda^2\sum_{k=0}^{\infty}\left[\frac{\lambda^k}{k!}\cdot e^{-\lambda}\right]$$

$$\sum_{k=0}^{\infty} k^2\cdot\left[\frac{\lambda^k}{k!}\cdot e^{-\lambda}\right] - 2\cdot\lambda^2 + \lambda^2\cdot e^{-\lambda}\cdot e^{\lambda}$$

$$\sum_{k=0}^{\infty} k^2\cdot\left[\frac{\lambda^k}{k!}\cdot e^{-\lambda}\right] - \lambda^2$$

Variance = Mean, Part 2

$$\sum_{k=0}^{\infty} k^2 \cdot \left[\frac{\lambda^k}{k!} \cdot e^{-\lambda} \right] - \lambda^2 = \sum_{k=0}^{\infty} \left[k \cdot (k-1) + k \right] \cdot \left[\frac{\lambda^k}{k!} \cdot e^{-\lambda} \right] - \lambda^2$$

$$\sum_{k=0}^{\infty} \left[k \cdot (k-1) \right] \cdot \left[\frac{\lambda^k}{k!} \cdot e^{-\lambda} \right] + \sum_{k=0}^{\infty} k \cdot \left[\frac{\lambda^k}{k!} \cdot e^{-\lambda} \right] - \lambda^2$$

$$\sum_{k=0}^{\infty} \left[k \cdot (k-1) \right] \cdot \left[\frac{\lambda^k}{k!} \cdot e^{-\lambda} \right] + \lambda - \lambda^2 = e^{-\lambda} \cdot \sum_{k=2}^{\infty} \left[k \cdot (k-1) \right] \cdot \left[\frac{\lambda^k}{k!} \right] + \lambda - \lambda^2$$

$$e^{-\lambda} \cdot \lambda^2 \cdot \sum_{k=2}^{\infty} \left[\frac{\lambda^{(k-2)}}{(k-2)!} \right] + \lambda - \lambda^2 = e^{-\lambda} \cdot \lambda^2 \cdot e^{\lambda} + \lambda - \lambda^2 = \lambda$$

References

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