Taylor Series

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Overview

- Physical systems are simulated in computers using numerical integration of the differential equations describing the changes over time
- The Taylor Series can be used to estimate a physical value at the next incremental time step
- Computation time can be reduced by using methods which drop higher order terms
- Accuracy can be increased by using smaller simulation time steps at the expense of computation time

Math Notation

- k! k factorial
  k • (k - 1) • (k - 2) • [...] • 3 • 2 • 1
  5! = 5 • 4 • 3 • 2 • 1 = 120
  0! = 1

- $f^{(k)}(t)$ k_{th} order derivative of a function of time
  $f^{(0)}(t) = f(t) = 0.5 \cdot t^2$
  $f^{(1)}(t) = f'(t) = 0.5 \cdot 2 \cdot t = t$
  $f^{(2)}(t) = f''(t) = 1$
  $f^{(3)}(t) = f'''(t) = 0$

Greek Math Symbols

- Σ Sigma: sum; add the elements that follow

$$\sum_{k=0}^{3} x^k = x^0 + x^1 + x^2 + x^3$$

- Δ Delta: difference; a step-wise change

$$f(x) = x^2$$

$$f(x + \Delta x) = (x + \Delta x)^2$$

$$x^2 + 2 \cdot x \cdot \Delta x + \Delta x^2$$

Taylor Series

$$f\left(t+\Delta t\right)=\sum_{k=0}^{\infty}\frac{\Delta t^{k}\cdot f^{(k)}\left(t\right)}{k!}$$

$$\frac{\Delta t^{0}\cdot f^{(0)}\left(t\right)}{0!}+\frac{\Delta t^{1}\cdot f^{(1)}\left(t\right)}{1!}+\frac{\Delta t^{2}\cdot f^{(2)}\left(t\right)}{2!}+\frac{\Delta t^{3}\cdot f^{(3)}\left(t\right)}{3!}+[\ldots]$$

$$f\left(t\right)+\Delta t\cdot f'\left(t\right)+\frac{\Delta t^{2}\cdot f''\left(t\right)}{2!}+\frac{\Delta t^{3}\cdot f'''\left(t\right)}{3!}+[\ldots]$$

Exponential Function

- An exponential function is linear in the sense that the first derivative is a function of the current value

$$f'(t) = \alpha \cdot f(t)$$

- Higher order derivatives are also a function of the current value

$$f''(t) = \alpha \cdot f'(t) = \alpha^2 \cdot f(t)$$
$$f'''(t) = \alpha \cdot f''(t) = \alpha^3 \cdot f(t)$$
$$f''''(t) = \alpha \cdot f'''(t) = \alpha^4 \cdot f(t)$$

Exponential Euler

$$f\left(t+\Delta t\right)=\sum_{k=0}^{\infty}\frac{\Delta t^{k}\cdot f^{(k)}\left(t\right)}{k!}$$

$$f\left(t\right)+\Delta t\cdot\alpha\cdot f\left(t\right)+\frac{\Delta t^{2}\cdot\alpha^{2}\cdot f\left(t\right)}{2!}+\frac{\Delta t^{3}\cdot\alpha^{3}\cdot f\left(t\right)}{3!}+[\ldots]$$

$$f\left(t\right)\cdot\left[1+\alpha\cdot\Delta t+\frac{\left[\alpha\cdot\Delta t\right]^{2}}{2!}+\frac{\left[\alpha\cdot\Delta t\right]^{3}}{3!}+[\ldots]\right]$$

$$f\left(t\right)\cdot\left[\sum_{k=0}^{\infty}\frac{\left[\alpha\cdot\Delta t\right]^{k}}{k!}\right]$$

$$f\left(t+\Delta t\right)=f\left(t\right)\cdot e^{\alpha\cdot\Delta t}$$

Euler's Method

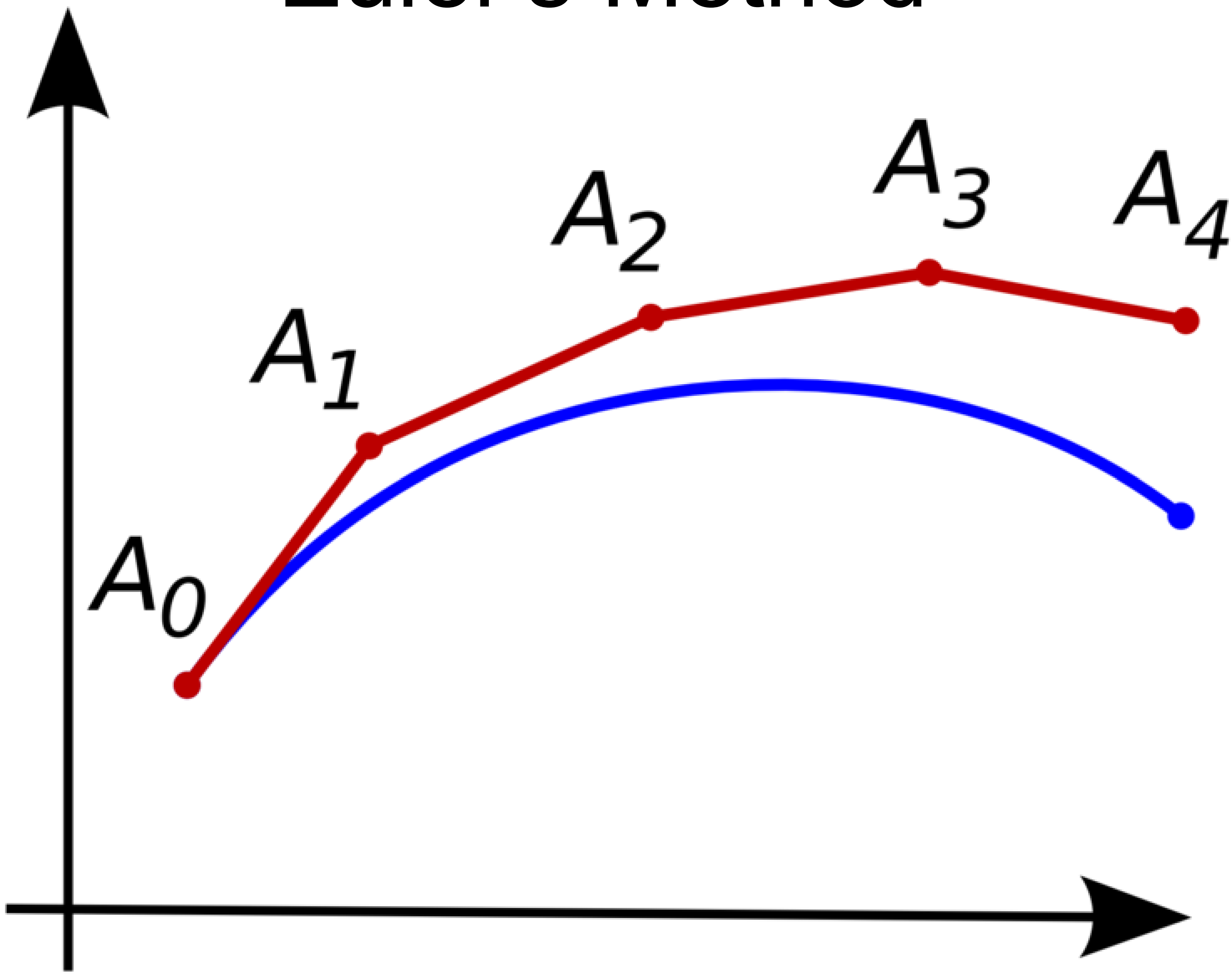
To reduce computation time, a simplifying approximation can be made for small values of Δt

$$f(t+\Delta t)=\sum_{k=0}^{\infty}\frac{\Delta t^{k}\cdot f^{(k)}(t)}{k!}$$

$$f(t)+\Delta t\cdot f'(t)+\frac{\Delta t^{2}\cdot f''(t)}{2!}+\frac{\Delta t^{3}\cdot f'''(t)}{3!}+[...]$$

$$f(t+\Delta t)\approx f(t)+\Delta t\cdot f'(t)$$

Euler's Method



Oleg Alexandrov, http://en.wikipedia.org/wiki/File:Euler_method.png

Midpoint Method

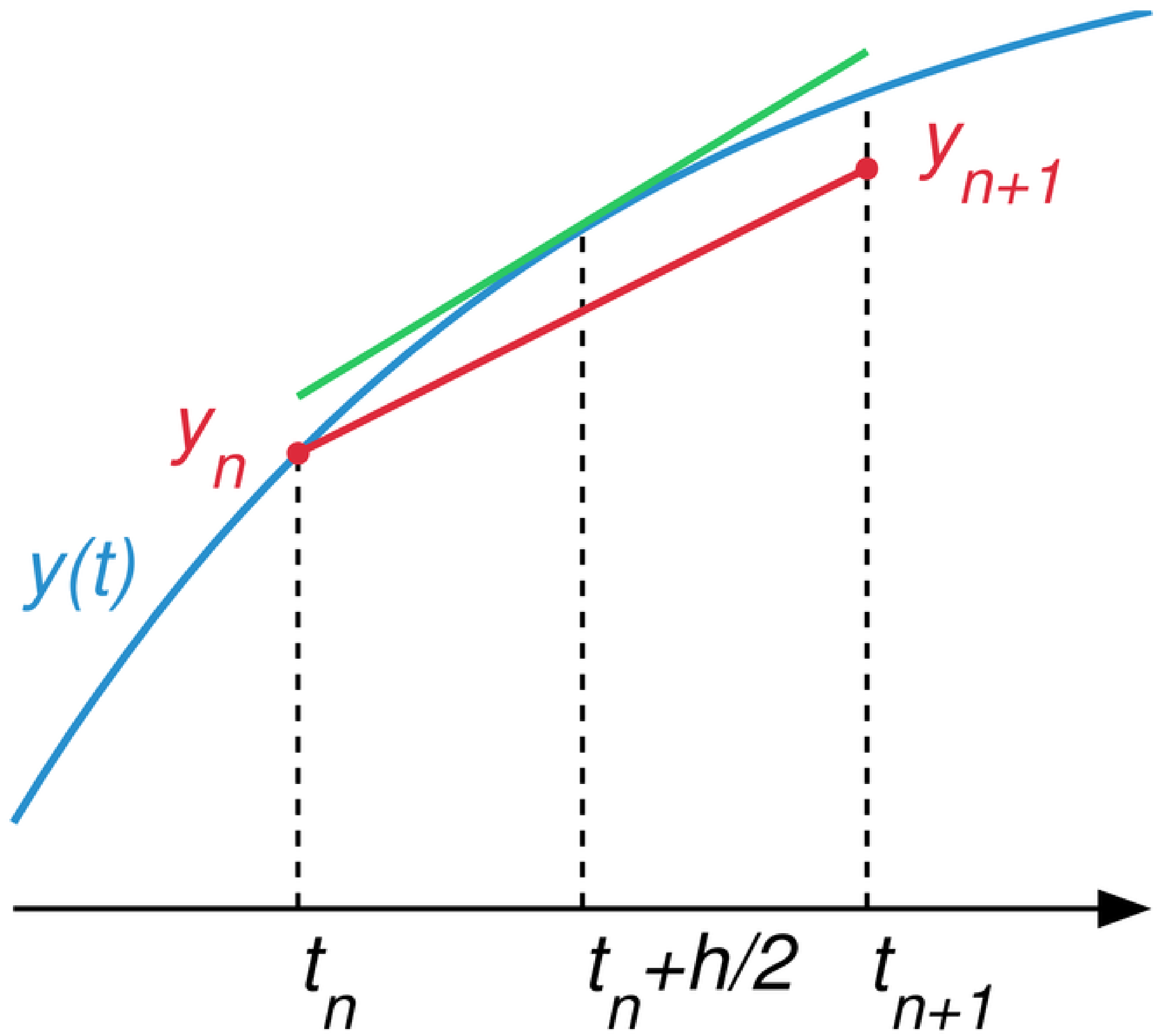
Use the estimated slope at the midpoint

$$f\left(t+\frac{\Delta t}{2}\right) \approx f_{midpoint\,estimate} = f(t) + \frac{\Delta t}{2} \cdot f'(t)$$

$$f'_{midpoint\,estimate} = f\left(t+\frac{\Delta t}{2},\, f_{midpoint\,estimate}\right)$$

$$f(t+\Delta t) \approx f(t) + \Delta t \cdot f'_{midpoint\,estimate}$$

Midpoint Method



Oleg Alexandrov, http://en.wikipedia.org/wiki/File:Midpoint_method_illustration.png

Midpoint Method for Exponential 1

$$f'(t) = \alpha \cdot f(t)$$

$$f(t+\Delta t) = f(t) \cdot e^{\alpha \cdot \Delta t}$$

$$f_{midpoint\ estimate} = f(t) + \frac{\Delta t}{2} \cdot [\alpha \cdot f(t)]$$

$$f_{midpoint\ estimate} = f(t) \cdot \left[1 + \frac{\Delta t}{2} \cdot \alpha\right]$$

$$f'_{midpoint\ estimate} = \alpha * f_{midpoint\ estimate}$$

$$f'_{midpoint\ estimate} = f(t) \cdot \left[\alpha + \frac{\Delta t}{2} \cdot \alpha^2\right]$$

Midpoint Method for Exponential 2

Equivalent to second order Taylor Series

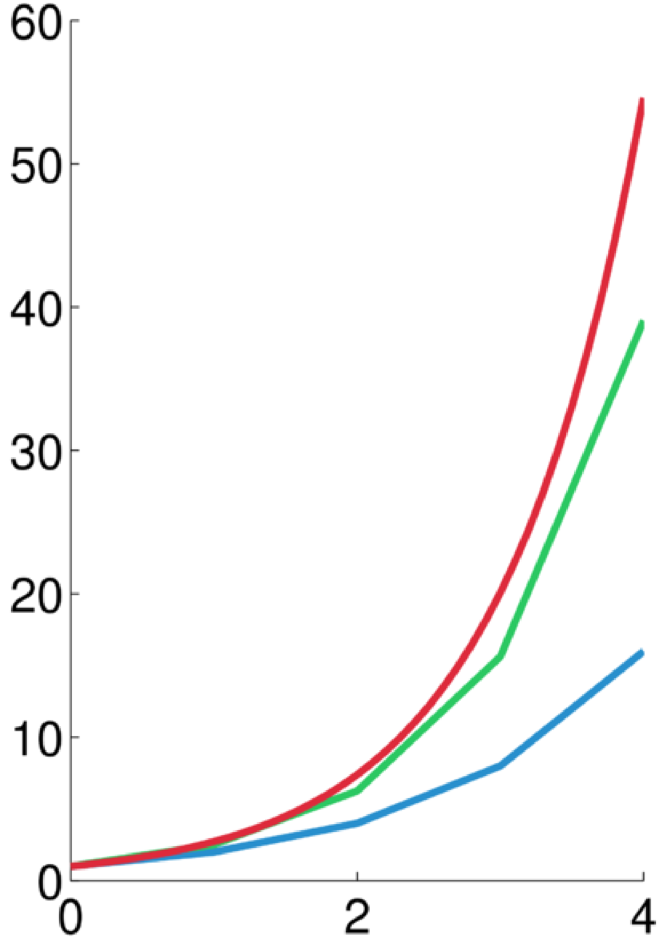
$$f'_{midpoint\ estimate} = f(t) \cdot \left[\alpha + \frac{\Delta t}{2} \cdot \alpha^2 \right]$$

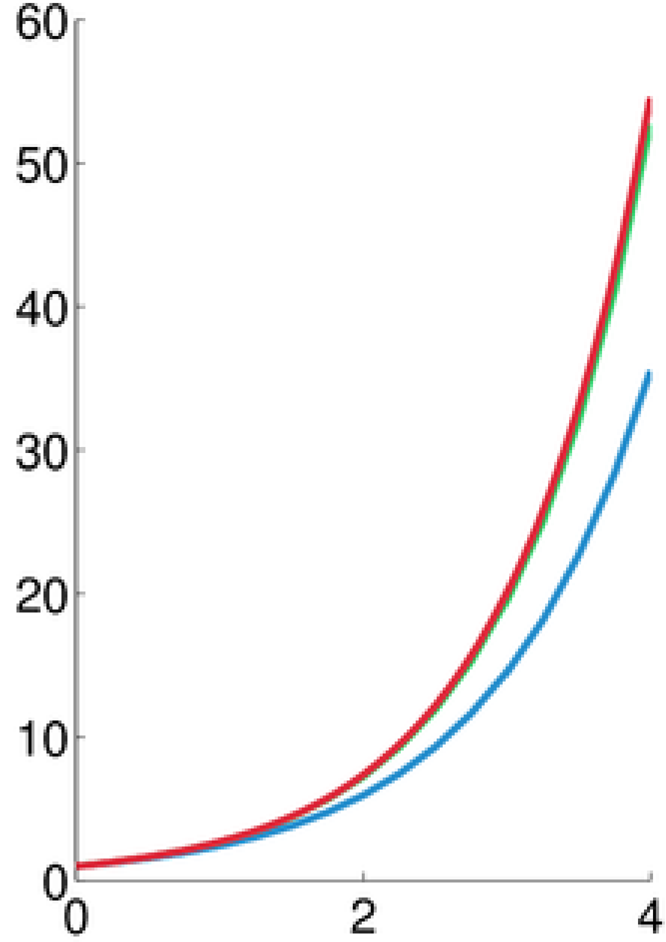
$$f(t + \Delta t) \approx f(t) + \Delta t \cdot f'_{midpoint\ estimate}$$

$$f(t) + \Delta t \cdot \left(f(t) \cdot \left[\alpha + \frac{\Delta t}{2} \cdot \alpha^2 \right] \right)$$

$$f(t + \Delta t) \approx f(t) \cdot \left[1 + \alpha \cdot \Delta t + \frac{[\alpha \cdot \Delta t]^2}{2} \right]$$

Small Δt





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References

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- Wikipedia, "Euler's Method"
- Wikipedia, "Midpoint Method"